American University of Beirut
Department of Computer Science
CMPS 211 – Discrete Mathematics – Fall 15/16
Assignment 2 Solution

Please solve the following exercises and submit **BEFORE 11:55 pm of Wednesday 7th, October.** Submit on Moodle.

Exercise 1 (10 points)

Determine whether each of these compound propositions following is satisfiable or not. When satisfiable, give the satisfying assignments for the variables, build truth table when it's not satifiable.

a) $(p \vee q \vee r \vee s) \wedge (\neg p \vee q \vee r \vee \neg s) \wedge (\neg p \vee \neg q \vee \neg r \vee s) \wedge (p \vee \neg q \vee r \vee \neg s)$
Satisfiable for p = true, q = true, r = false
b) $(p \vee q \vee \neg s \vee \neg r) \wedge (p \vee \neg q \vee s \vee r) \wedge (p \vee q \vee s \vee r) \wedge (\neg p \vee q \vee s \vee r) \wedge (p \vee q \vee r \vee s) \wedge (\neg p \vee \neg q \vee \neg r \vee \neg s)$
Satisfiable for p = true, q = true, r = false
c) $(p \vee q) \wedge (\neg p \vee \neg q) \wedge (p \vee \neg q \vee r)$
Satisfiable for p = true, q = false

Exercise 2 (10 points)

Determine whether the following are tautologies **without** using truth tables

a) $p \wedge q \rightarrow p \vee q$
$p \wedge q \rightarrow p \vee q$
$\Leftrightarrow \neg(p \wedge q) \vee (p \vee q)$
$\Leftrightarrow (\neg p \vee \neg q) \vee (p \vee q)$
$\Leftrightarrow (\neg p \vee p) \vee (\neg q \vee q)$
$\Leftrightarrow T \vee T$
$\Rightarrow$ Tautology
b) $[(q \rightarrow p) \wedge (r \wedge p) \wedge (p \rightarrow q)] \rightarrow p$
$[(q \rightarrow p) \wedge (r \wedge p) \wedge (p \rightarrow q)] \rightarrow p$
$\Leftrightarrow [(\neg q \vee p) \wedge (r \wedge p) \wedge (\neg p \vee q)] \rightarrow p$
$\Leftrightarrow \neg[(\neg q \vee p) \wedge (r \wedge p) \wedge (\neg p \vee q)] \vee p$
$\Leftrightarrow [\neg (\neg q \vee p) \vee \neg (r \wedge p) \vee \neg (\neg p \vee q)] \vee p$
$\Leftrightarrow (q \wedge p) \vee (\neg r \vee \neg p) \vee (p \wedge \neg q) \vee p$
$\Leftrightarrow (q \wedge p) \vee (p \wedge \neg q) \vee \neg r \vee (\neg p \vee p)$
$\Leftrightarrow (q \wedge p) \vee (p \wedge \neg q) \vee \neg r \vee T$
$\Rightarrow$ Tautology
c) $(p \rightarrow q) \leftrightarrow [\neg p \vee (p \wedge q)]$
$(p \rightarrow q) \leftrightarrow [\neg p \vee (p \wedge q)]$
$\Leftrightarrow (\neg p \vee q)] \leftrightarrow [\neg p \vee (p \wedge q)]$
$\Leftrightarrow (\neg p \vee q)] \rightarrow [\neg p \vee (p \wedge q)] \wedge [\neg p \vee (p \wedge q)] \rightarrow (\neg p \vee q)]$

$\Leftrightarrow \neg\,(\neg p \vee q)] \vee [\neg p \vee (p \wedge q)] \wedge \neg\,[\neg p \vee (p \wedge q)] \vee (\neg p \vee q)]$
$\Leftrightarrow (p \wedge \neg q)] \vee [\neg p \vee (p \wedge q)] \wedge [\neg p \wedge \neg\,(p \wedge q)] \vee (\neg p \vee q)]$
$\Leftrightarrow (p \wedge \neg q)] \vee [\,q\,] \wedge [\neg p \wedge (\neg p \vee \neg q)] \vee (\neg p \vee q)]$
$\Leftrightarrow (p \wedge \neg q)] \vee [\,q\,] \wedge [\neg p \wedge (\neg p \vee \neg q)] \vee (\neg p \vee q)]$
$\Leftrightarrow (p \vee q) \wedge (\neg q \vee q) \wedge [(\neg p \wedge \neg p) \vee (\neg p \wedge \neg q)] \vee (\neg p \vee q)]$
$\Leftrightarrow (p \vee q) \wedge T \wedge [\neg p \vee (\neg p \vee q)]$
$\Leftrightarrow (p \vee q) \wedge (\neg p \vee q)$
$\Leftrightarrow q$
but q can be either true or false, then its not a tautology

Exercise 3 (15 points)

Consider the logical operations *NAND*. The proposition *p NAND q* is false when p and q are both true, and true otherwise. The propositions p *NAND* q is denoted by p **|** q.

a) Construct a truth table for the logical operator *NAND*.

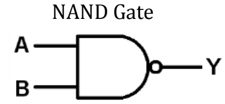
p	q	$p \mid q$
T	T	F
T	F	T
F	T	T
F	F	T

b) Show that $p \mid q$ is logically equivalent to $\neg(p \wedge q)$.

p	q	$p \wedge q$	$\neg(p \wedge q)$	$p \mid q$	$\neg(p \wedge q) \leftrightarrow p \mid q$
T	T	T	F	F	T
T	F	F	T	T	T
F	T	F	T	T	T
F	F	F	T	T	T

c) NAND gates are universal gates, meaning that you can create any other gate using NAND gates only, combined in different ways. Using NAND gates only draw circuits that can operate in equivalence to:

NAND Gate



i. Not gate
$\neg p \Leftrightarrow p \mid p$

ii. AND gate
$(p \wedge q)$
$\Leftrightarrow \neg\,(p \mid q)$
$\Leftrightarrow (p \mid q) \mid (p \mid q)$

iii. OR gate
$(p \vee q)$

$\Leftrightarrow$ ¬ (¬p ∧¬q)
$\Leftrightarrow$ ¬[(p | p) ∧ (q | q)]
$\Leftrightarrow$ ¬[{(p | p) | (q | q)} | {(p | p) | (q | q)}]
$\Leftrightarrow$ ¬[{(p | p) | (q | q)} | {(p | p) | (q | q)}]
$\Leftrightarrow$ [{(p | p) | (q | q)} | {(p | p) | (q | q)}] | [{(p | p) | (q | q)} | {(p | p) | (q | q)}]
Also equivalent to ¬ p | ¬q

iv. NOR gate (p NOR q is true when both p and q are false, false otherwise)
(p ↓ q)
$\Leftrightarrow$ ¬(p ∨ q)
$\Leftrightarrow$ (¬p ∧¬q)
$\Leftrightarrow$ (¬p | ¬q) | (¬p | ¬q)
$\Leftrightarrow$ ((p | p) | (q | q)) | ((p | p) | (q | q))
Also equivalent to ¬(¬p | ¬q)

Exercise 4 (10 points)

Determine whether $\forall x(P(x) \leftrightarrow Q(x))$ and $\forall x\, P(x) \leftrightarrow \forall x Q(x)$ are logically equivalent. Justify your answer.

No, they aren't logically equivalent.
Assume $P(x)$ = x < 0, and $Q(x)$ = x ≥ 0, and the domain of x is all integers;

- $\forall x\, P(x)$ is false, and $\forall x\, Q(x)$is also false, then $\forall x\, P(x) \leftrightarrow \forall x Q(x)$ is true
- but P(1) is false and Q(1) is true, so $P(1) \leftrightarrow Q(1)$ is false, then $\forall x(P(x) \leftrightarrow Q(x))$ is false

Then $\forall x\, P(x) \leftrightarrow \forall x Q(x)$ is not logically equivalent to $\forall x(P(x) \leftrightarrow Q(x))$

Exercise 5 (10 points)

Translate the given statement into propositional logic using the propositions provided:

You can upgrade your operating system only if you have a 32-bit processor running at 1 GHz or faster, at least 1 GB RAM, and 16 GB free hard disk space, or a 64-bit processor running at 2 GHz or faster, at least 2 GB RAM, and at least 32 GB free hard disk space.

Express you answer in terms of:

- *u*: "You can upgrade your operating system"
- *b*32: "You have a 32-bit processor"
- *b*64: "You have a 64-bit processor"

- $g1$: "Your processor runs at 1 GHz or faster"
- $g2$: "Your processor runs at 2 GHz or faster"
- $r1$: "Your processor has at least 1 GB RAM"
- $r2$: "Your processor has at least 2 GB RAM"
- $h16$: "You have at least 16 GB free hard disk space"
- $h32$: "You have at least 32 GB free hard disk space"

$u \rightarrow (b32 \wedge g1 \wedge r1 \wedge h16) \vee (b64 \wedge g2 \wedge r2 \wedge g32)$

Exercise 6 (10 points)

Let

- $P(x)$ = "x is a clear explanation",
- $Q(x)$ = "x is satisfactory"
- $R(x)$ ="x is an excuse"

where the domain for x consists of all English text.

Express each of these statements using quantifiers, logical connectives, and $P(x)$, $Q(x)$, and $R(x)$. Then express their negations in English and using quantifiers.

a) All clear explanations are satisfactory.
   $\forall x\, (P(x) \rightarrow Q(x))$
b) Some excuses are unsatisfactory.
   $\exists x\, R(x) \wedge \neg Q(x)$
c) Some excuses are not clear explanations.
   $\exists x\, \neg P(x) \wedge R(x)$
d) Does (c) follow from (a) and (b)?
   Yes it follows. Since being unsatisfactory is guaranteed in (b), then being unclear is deduced from (a)

1. $\exists x\, R(x) \wedge \neg Q(x)$	Premise
2. $R(c) \wedge \neg Q(c)$	Existential Instantiation from 1
3. $\neg Q(c)$	Simplification from 2
4. $\forall x\, (P(x) \rightarrow Q(x))$	Premise
5. $\forall x\, (\neg Q(x) \rightarrow \neg P(x))$	Contraposition from 4
6. $\neg Q(c) \rightarrow \neg P(c)$	Universal Instantiation from 5
7. $\neg P(c)$	Modus Pollens from 3 and 6
8. $R(c)$	Simplification from 2
9. $R(c) \wedge \neg P(c)$	Conjugation from 7 and 8

10. ∃x ¬P(x) ∧ R(x) Existential Generalization from 9

Exercise 7 (10 points)

Suppose the propositional function P(x) = x^2 -5 > 12 → x^3 + 9 > -10, and the domain that consists of {−10, -4, 0, 4}. Express these statements without using quantifiers, instead using only negations, disjunctions, and conjunctions, and evaluate each statement.

a) ∃xP(x)
P(-10) ∨ P(-4) ∨ P(0) ∨ P(4)
↔ (95 > -10 → -991 > -10) ∨ (11 > -10 → -55>-10) ∨ …∨
↔(T → F) ∨ (F → F) ∨..∨
↔ F ∨ T ∨ … ∨…
↔ T

b) ∀xP(x)
P(-10) ∧ P(-4) ∧ P(0) ∧ P(4)
False, since P(-4) is false

c) ∀x((x≠10)→P(x))
x ≠ 10 is always true in our domain, then ∀x((x≠10)→P(x)) ↔ ∀xP(x) in part c, then false

d) ∃x(¬P(x)) ∧ ∀x((x>4)→P(x))
We know that P(-10) is only false, and all the rest are true; since we need ¬P(x), then we can plug in -10 for x in ∃x(¬P(x)), instead of enumerating all the domain and combine with disjunctions
→ P(-10) ∧ [(-10 > 4 →P(-10)) ∧ (-4 > 4 →P(-4)) ∧ (0 > 4 →P(0)) ∧ (4 > 4 →P(4))
↔T ∧ (F → T ∧ F → F ∧ F → F ∧ F → F)
↔ T ∧ T ∧ T ∧ T ∧ T
↔ T

Exercise 8 (15 points)

Translate in two ways each of these statements into logical expressions using predicates, quantifiers, and logical connectives. First, let the domain consist of all transportations tools (Cars, Motorcycle, Buses, Planes, Trains, Horses, etc..) and the second be cars.

Let C(x) = "x is a car", where x belongs to all transportation tools
Let O(p, x) = "person p owns object x", where domain of p belong to all people

a) Some cars run on solar power
Let $S(x)$ = "x runs on solar power" where x belongs to all transportation tools
   1. $\exists x\ C(x) \land S(x)$
   2. $\exists x\ S(x)$

b) All people own at least one car
   1. $\forall p\ \exists x\ (C(x) \land O(p,x))$
   2. $\forall p\ \exists x\ O(p,x)$

c) Some people own 2 (or more) cars
   1. $\exists p\ \exists x\ \exists y\ x \neq y \land C(x) \land C(y) \land O(p,x) \land O(p,y)$
   2. $\exists p\ \exists x\ \exists y\ x \neq y \land O(p,x) \land O(p,y)$

d) Some people own no car.
   1. $\exists p\ \forall x\ (\neg C(x) \lor \neg O(p,x))$
   2. $\exists p\ \forall x\ \neg O(p,x)$

e) Some cars are faster than all non-cars (*other transport tools)*
Let $F(x,y)$ = "x is faster than y"
   1. $\exists x\ \forall y\ (C(x) \land \neg C(y) \rightarrow \text{Faster}(x, y))$
   2. Can't be expressed since the domain only contain cars [unless we create a new domain for non-cars]…

f) Exactly 2 cars crashed last night
Let Crashed(x) = "x crashed last night"
   1. *2 more:* $\exists x\ \exists y\ \forall z\ (C(x) \land C(y) \land \text{Crashed}(x) \land \text{Crashed}(y) \land x \neq y)$
   Exactly 2: $\exists x\ \exists y\ \forall z\ [C(x) \land C(y) \land \text{Crashed}(x) \land \text{Crashed}(y) \land x \neq y \land (x = z \lor y = z \lor \neg C(z) \lor \neg \text{Crashed}(z))]$
   2. $\exists x\ \exists y\ \forall z\ [\text{Crashed}(x) \land \text{Crashed}(y) \land x \neq y \land (x = z \lor y = z \lor \neg \text{Crashed}(z))]$

Exercise 9 (10 points)

Express these propositions and their negations using quantifiers, and in English.

a) There is a soccer player who didn't score any goal.
Let $S(x)$: "player x score at least a goal"
Original: $\exists x\ \neg S(x)$
English Negation: All score players scored at least one goal
Propositional Negation: $\forall x\ S(x)$

b) Every professor taught all courses in his department
Let
- $T(p, c)$ = "Prof. p taught course c",
- $O(c, d)$ = "Course c is offered in dept. d"
- $M(p, d)$ = "Prof p is a member of dept d"

where domain of p is professors, and domain of c is courses, and d is departments

Original: ∀p ∀d ∀c [O(c,d) ∧ M(p,d) → T(p,c)]

English Negation: Some professors didn't teach a course in their department

Propositional Negation: ∃p ∃d ∃c [O(c, d) ∧ M(p, d) ∧ ¬T(p, c)]

For simplicity you may assume domain of c is courses in department d which the professor is a member of, and then remove the departments domains

c) Some taxi drivers have passed through at least one street in all area of Lebanon.
Let P(t, s, a) = "Taxi driver *t* passed through street *s* in area *a* in Lebanon", where the domains are obvious

Original: ∃p ∀a ∃s P(t, s, a)

English Negation: All taxi drivers have a street in some area in which they didn't pass through

Propositional Negation: ∀p ∃a ∀s ¬P(t, s, a)

d) Each lab has a computer that was never used by any student.
Let U(s, c ,l) = "Student s used computer c in lab l", where the domains are also obvious

Original: ∀l ∃c ∀s ¬U(s, c, l)

English Negation: All computers in some labs where used by students *OR* Some labs has no computers which were never used by students

Propositional Negation: ∃l ∀c ∃s U(s, c, l)

Exercise 10 (20 points)

Let F(x, y) be the statement "person x is a Facebook friend of person y", and K(x, y) "person x knows person y", S(x, z): "x is a student at university y", where the domain x and y is people, and that of z is universities. Express each of those statements and their negations in English/using quantifiers.

Note that F and K are commutative functions, meaning that F(x, y) = F(y, x) and K(x, y) = K(y, x)

a) All People who are Facebook friends with Joe know Joe
∀x F(x, Joe) → K(x, Joe)

b) Some people know everyone but isn't Facebook friend with anyone
∃x ∀y K(x, y) ∧ ¬F(x, y)

c) All university students are Facebook friend with each other
Let U(x) = "x is a university student"
∀x∀y [U(x) ∧ U(y) → F(x, y)]
OR if you don't need to create U(x):
∀x ∀z1 ∀y ∀z2 [(S(x, z1) ∧ S(y, z2) ∧ F(x,y)) ∨ (¬ S(x, z1) ∨ ¬ S(y, z2)]

d) ∀x ∃y [(S(x, AUB) → x≠y ∧ S(y, AUB) ∧ ¬K(x, y)]
Every AUB student doesn't know at least one other AUB student

e) ∃z ∃x ∀y S(x,z) ∧ ¬F(x,y)
Some students in some universities are not Facebook friends with anyone

f) All Facebook friends aren't in any university
∀x∀y∀z1∀z2 F(x, y) → (¬ S(x, z1) ∧ ¬S(y,z2))

Exercise 11 (10 points)

Show that $\forall xP(x) \lor \forall xQ(x)$ and $\forall x\forall y(P(x) \lor Q(y))$, where all quantifiers have the same nonempty domain, are logically equivalent. (The new variable *y* is used to combine the quantifications correctly.)

To prove this true, we need to prove:

1. $\forall xP(x) \lor \forall xQ(x) \rightarrow \forall x\forall y(P(x) \lor Q(y))$
2. $\forall x\forall y(P(x) \lor Q(y)) \rightarrow \forall xP(x) \lor \forall xQ(x)$

1. $\forall xP(x) \lor \forall xQ(x)$ Premise
2. $P(c) \lor \forall xQ(x)$ U.I from 1
3. $P(c) \lor Q(d)$ U.I from 2
4. $\forall xP(x) \lor Q(d)$ U.G from 3
5. $\forall x\forall y(P(x) \lor Q(y)$ U.G from 4

Then $\forall xP(x) \lor \forall xQ(x) \rightarrow \forall x\forall y(P(x) \lor Q(y)$ is true

1. ∀xP(x) ∨ ∀xQ(x) Premise
2. P(c) ∨ ∀xQ(x) U.I from 1
3. P(c) ∨ Q(d) U.I from 2
4. ∀x(P(x) ∨ Q(d)) U.G from 3
5. ∀x∀y(P(x) ∨ Q(y) U.G from 4

Then $\forall xP(x) \lor \forall xQ(x) \rightarrow \forall x\forall y(P(x) \lor Q(y)$ is true

Then they are logically equivalent